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Scaling laws for fully developed turbulent shear flows. Part 2. Processing of experimental data

By G. I. BARENBLATT† AND V. M. PROSTOKISHIN

P. P. Shirshov Institute of Oceanology, Russian Academy of Sciences, Moscow, 117218, Russia

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In Part 1 of this work (Barenblatt 1993) a non-universal scaling law (depending on the Reynolds number) for the mean velocity distribution in fully developed turbulent shear flow was proposed, together with the corresponding skin friction law. The universal logarithmic law was also discussed and it was shown that it can be understood, in fact, as an asymptotic branch of the envelope of the curves corresponding to the scaling law.

Here in Part 2 the comparisons with experimental data are presented in detail. The whole set of classic Nikuradze (1932) data, concerning both velocity distribution and skin friction, was chosen for comparison. The instructive coincidence of predictions with experimental data suggests the conclusion that the influence of molecular viscosity within the main body of fully developed turbulent shear flows remains essential, even at very large Reynolds numbers. Meanwhile, some incompleteness of the experimental data presented in the work of Nikuradze (1932) is noticed, namely the lack of data in the range of parameters where the difference between scaling law and universal logarithmic law predictions should be the largest.

1. Introduction

We shall speak, for definiteness, about flows in tubes because the data we use here for comparison with predictions are relevant to these flows. Both the universal logarithmic law and the scaling (power-type) law for the mean velocity distributions $u(y)$ in cylindrical tubes have equally rigorous foundations, based, however, on essentially different assumptions, as explained in Part 1 (Barenblatt 1993). The assumption leading to the logarithmic law is that the velocity gradient within the main body of the flow is completely independent of the molecular viscosity. The assumption leading to the scaling (power-type) law is that such dependence is retained at arbitrarily large Reynolds number, although it assumes an asymptotic form, i.e. scaling law, with invariance under a certain renormalization group, called *incomplete similarity* (Barenblatt 1979). Therefore these two laws proposed for the same quantity (the mean velocity distribution in the intermediate range of distance from the wall), are *qualitatively* different.

In Part 1 a scaling law was proposed for the mean velocity gradient, $\partial_y u$:

$$\partial_y u = (u_*/y)\,\Phi(\eta, Re), \quad \Phi = \alpha C \eta^{\alpha}, \tag{1}$$

where u_* is the dynamic or friction velocity, y the distance from the wall, $\eta = u_* y/\nu$, $Re = \bar{u} d/\nu$ the flow Reynolds number, ν the kinematic viscosity of the fluid, $\bar{u}$ the

† Present address: Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK.

mean velocity across a section, d the tube diameter, and C, α are constants depending on Re in a certain specific way.

We will present here in detail the checking of the basic conjecture, proposed in Part 1, that

$$\alpha = 3/2 \ln Re. \tag{2}$$

Verification confirms this conjecture to a high degree of accuracy. Furthermore, this checking allows us to calculate an accurate approximation to the coefficient C as a function of Reynolds number. Both the relation (2) and this approximation allowed us to obtain a proposed form and quasi-universal representation of the basic scaling law.

In the plane ($\phi = u/u_*$, $\ln \eta$), the scaling law is represented by a family of curves, each of them corresponding to its Reynolds number as a parameter. The family possesses an envelope. It was necessary to check whether the experimental points settle down close to the envelope or deviate from it. Furthermore, the corresponding skin friction law was also derived, so that it could also be compared with the experimental data, giving an independent verification of the scaling law.

Only Nikuradze's (1932) data concerning turbulent flows in cylindrical tubes were chosen here for the comparison with the predictions. This represents a large variety of data most carefully obtained under the guidance and observation of L. Prandtl. Up to the present time, this set of data is unrivalled in its completeness of presentation, and embraces an unsurpassed range of Reynolds number. Although, as will be seen later, this set of data has an essential shortcoming (the data cover an incomplete range of parameters) it forms a necessary first step from which to begin checking any newly proposed law for the velocity distribution and skin friction. Further checking will depend on future experiments to cover the missing parameter ranges.

2. Processing of velocity distribution data

2.1. *Verification of basic conjecture*

As in Part 1, equation (1) can be rewritten as

$$\partial_\eta \phi = (1/\eta)\, \Phi(\eta, Re) = \alpha C \eta^{\alpha-1}, \tag{3}$$

whence, by integration,

$$\phi = C\eta^{\alpha} + D. \tag{4}$$

We emphasize that *a priori* the integration constant D cannot be taken equal to zero. Indeed, the boundary condition $\phi = 0$ at $\eta = 0$ cannot be used: the point $\eta = 0$ lies outside the intermediate interval of distances where the asymptotic law (3) is assumed to be valid. Therefore in our primary processing, the coefficient D was taken into account and only later was it neglected, because it was shown that its contribution lies within the experimental scatter.

Therefore, the relation

$$\phi = C\eta^{\alpha}(1 + D/C\eta^{\alpha}) \tag{5}$$

was proposed for checking with $\alpha = 3/(2 \ln Re)$. The parameter α is a small number of the order of one tenth in the experiments. A rather severe procedure for checking this relation was chosen: all the values of ϕ from all sixteen series of table 3 from Nikuradze (1932) were raised to the large (and, we emphasize, preassigned, and not adjusted each time) degree $(2 \ln Re)/3$, and were plotted on graphs against η. The results are presented in figure 1(*a*–*e*). It is seen clearly that in the intermediate

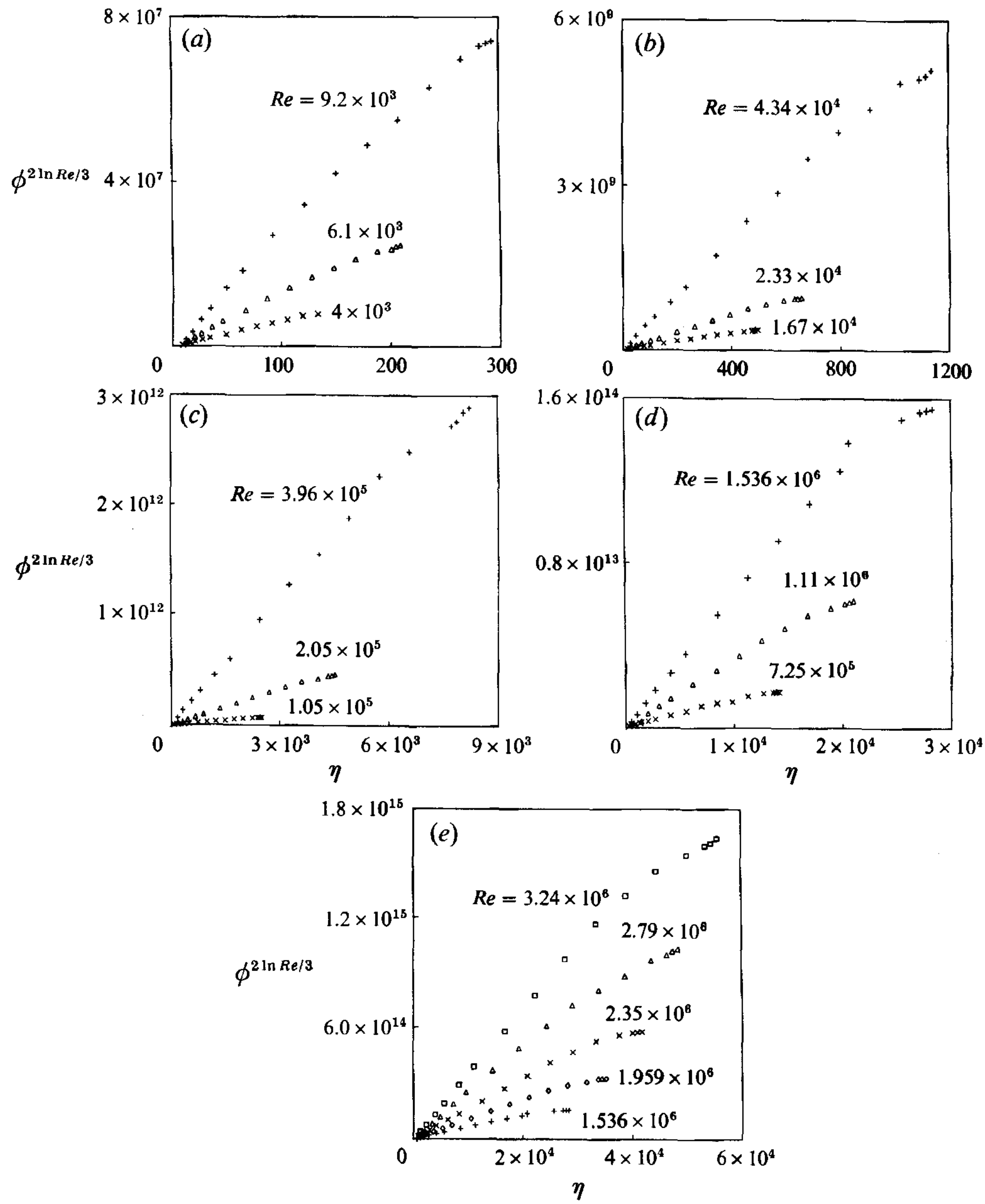

FIGURE 1. (*a–e*). The graphs of $\phi^{(2\ln Re)/3}(\eta)$ reveal straight lines in an intermediate interval of values of η, for various Re values as shown.

interval of η, straight lines have indeed appeared, more and more clearly as Re increases, over a larger and larger η-interval including the origin. This strongly indicates that, according to our basic conjecture, the power $(2\ln Re)/3$ is correctly chosen, because even a relatively small error would transform the straight line segments into curves with positive or negative curvature. It is also seen that the contribution of the term $D/C\eta^{\alpha}$ in the intermediate interval of our interest is negligibly small – we recall that the intermediate asymptotic law (1) is invalid within the viscous layer close to the wall where η is rather small, and within a certain region

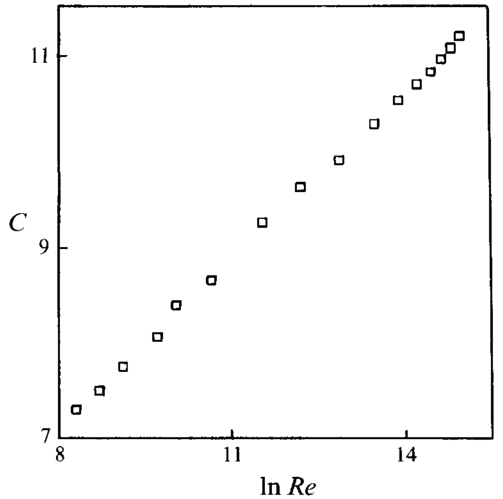

FIGURE 2. The function $C(\ln Re)$ obtained by the processing of experimental data.

near the tube axis. As can be seen, the straight lines shown in figure 1 (*a*) ($Re = 4 \times 10^3$; 6.1×10^3; 9.2×10^3), corresponding to minimal Reynolds numbers in the series, are less accurate than in figure 1 (*b–e*). The reason for that seems to be natural: at such Reynolds numbers the turbulent flow is not yet fully developed and so, according to Prandtl's idea (see Nikuradze 1932) special additional turbulizers were installed. The inaccuracy lies within experimental scatter but we also notice that the artificial character of the imposed additional turbulence could have had some influence here too.

2.2. *Determining the pre-power constant*

The values of the slopes of each straight line in figure 1 (*a–e*) were determined by a standard least-squares method, and each of the values so obtained was raised to the power $\alpha = 3/(2 \ln Re)$. Thus the value of C for each Re was determined, and C was obtained as a function of $\ln Re$, presented in figure 2.

This function is obviously close to a linear one, and so it was approximated by a straight line regression,

$$C = A \ln Re + B. \tag{6}$$

The coefficients of the regression (6) were also estimated by the method of least squares over all sixteen series of experiments. Leaving aside experimental errors, these coefficients are influenced by some arbitrariness in discarding the experimental points near the wall and near the tube axis. This influence is illustrated by table 1, where the values of the coefficients estimated by discarding various numbers of points near the wall and near the axis are presented.

For convenience, we used the following representation of the coefficient C:

$$C = \frac{1}{\surd 3} \ln Re + \frac{5}{2}. \tag{7}$$

The reason for that was twofold: firstly, $1/\surd 3 \approx 0.577$ coincides deeply within the experimental scatter with the empirical value 0.578 (see table 1), and secondly it was found that the hypothesis that $A = 1/\surd 3$ exactly greatly simplifies the subsequent manipulation of formulae.

Number of experiment	The range k_1–k_2	A	B
1	2–15	0.577	2.49
2	3–14	0.578	2.50
3	4–13	0.579	2.51
4	4–15	0.577	2.48
5	4–14	0.578	2.49
6	5–11	0.578	2.53
Mean value		0.578 ± 0.0017	2.50 ± 0.016

TABLE 1. The coefficients A, B in relation (6). Note: k_1 is the number of the first non-discarded point near the wall; k_2 is the number of the last non-discarded point near the axis. The total number of points in each series is 16.

2.3. *The quasi-universal form of the scaling law*

According to (2) and (7), the scaling law can be represented in the final form

$$\phi = \left(\frac{1}{\sqrt{3}} \ln Re + \tfrac{5}{2}\right) \eta^{3/2 \ln Re}, \tag{8}$$

or

$$\phi = \left(\frac{\sqrt{3}+5\alpha}{2\alpha}\right) \eta^{\alpha}, \quad \alpha = \frac{3}{2 \ln Re}. \tag{9}$$

A quasi-universal form of this law can be obtained by defining a new function ψ,

$$\psi \equiv \frac{1}{\alpha} \ln \frac{2\alpha\phi}{\sqrt{3}+5\alpha} = \ln \eta. \tag{10}$$

This form is convenient because, instead of a family of curves, a single curve in reduced variables is obtained. Indeed, according to (10), the experimental points in the $(\psi, \ln \eta)$-plane should settle down on the bisectrix of the first quadrant, and this is in fact so (see figure 3). Accuracy (except for a very few points, from a total of 256, corresponding mainly to small values of η where the asymptotic behaviour is not expected to hold) is in favour of the proposed scaling law. An additional convenience of the quasi-universal representation (10) lies in the possibility of comparing the accuracy with the graph of the universal logarithmic law, where $\ln \eta$ is also usually plotted on the abscissa axis. At small η a systematic deviation from the bisectrix is observed. This is natural because these points are outside the main body of the flow.

2.4. *The scaling laws in the $(\phi, \ln \eta)$-plane and their envelope: comparison with the universal logarithmic law*

Consider, in the $(\phi, \ln \eta)$-plane, the three experimental series with Reynolds numbers differing approximately by an order of magnitude: (i) $Re = 1.67 \times 10^4$, (ii) $Re = 2.05 \times 10^5$, and (iii) $Re = 3.24 \times 10^6$. In figure 4 all the experimental points of these series are presented together with the scaling curves (9) corresponding to their Reynolds numbers (curves i, ii, iii), the universal logarithmic law straight line (curve iv), and the envelope of the family of scaling law curves (curve v). We see that there appears to be a discernible advantage in favour of the curves of the scaling laws: there is a systematic, albeit small, deviation from the universal (independent of Reynolds number) curves (iv) and (v). Note that the small but systematic deviations

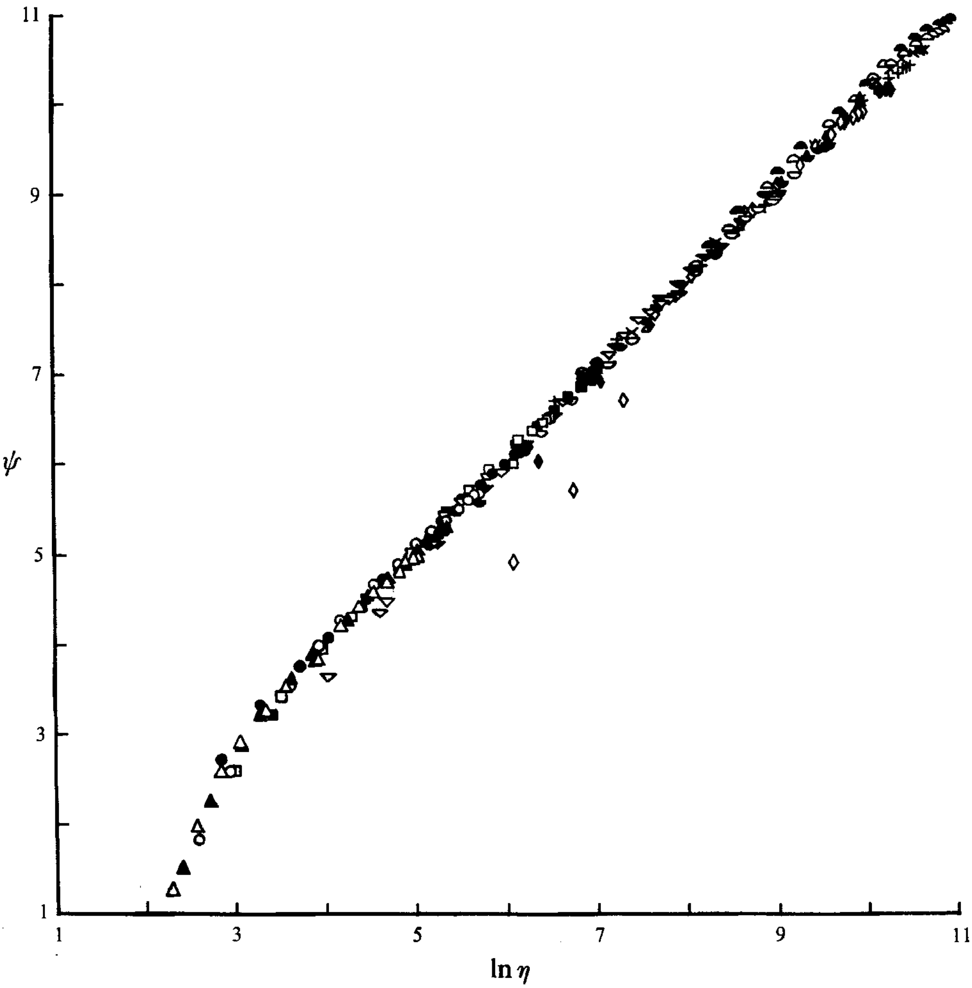

FIGURE 3. The experimental points in reduced coordinates $(\psi, \ln\eta)$ settle down, for large η, close to the bisectrix of the first quadrant, confirming the quasi-universal form of the scaling law. $\triangle$, $Re = 4\times10^3$; $\blacktriangle$, $Re = 6.1\times10^3$; $\bigcirc$, $Re = 9.2\times10^3$; $\bullet$, $Re = 1.67\times10^4$; $\square$, $Re = 2.33\times10^4$; $\blacksquare$, $Re = 4.34\times10^4$; $\triangledown$, $Re = 1.05\times10^5$; $\blacktriangledown$, $Re = 2.05\times10^5$; $\smile$, $Re = 3.96\times10^5$; $\smile$, $Re = 7.25\times10^5$; $\diamond$, $Re = 1.11\times10^6$; $\blacklozenge$, $Re = 1.536\times10^6$; $+$, $Re = 1.959\times10^6$; $\times$, $Re = 2.35\times10^6$; $\frown$, $Re = 2.79\times10^6$; $\frown$, $Re = 3.24\times10^6$.

from the universal logarithmic law, in favour of power-type laws with empirically fitted coefficients, were noted by Nikuradze himself. It is interesting to offer some quantitative estimates: the relative mean-square error σ_s for the scaling law (9) and the corresponding mean-square error σ_u for the universal logarithmic law were calculated. The following figures were obtained: $\sigma_u = 2.1\times10^{-2}$, $\sigma_s = 1.1\times10^{-2}$.

In table 2, the relative mean-square errors are given for the universal law, σ_u, and for the proposed scaling law (9), σ_s. The values are calculated for all sixteen experimental series. We do not want to emphasize here the difference in favour of the proposed scaling law: this difference, in our opinion, is not significant.

We do, however, wish to stress another circumstance, namely that Nikuradze (1932) also worked with a large tube, with $d = 10$ cm. He obtained the drag results plotted in figure 5. The velocity data for this tube unfortunately were not available.

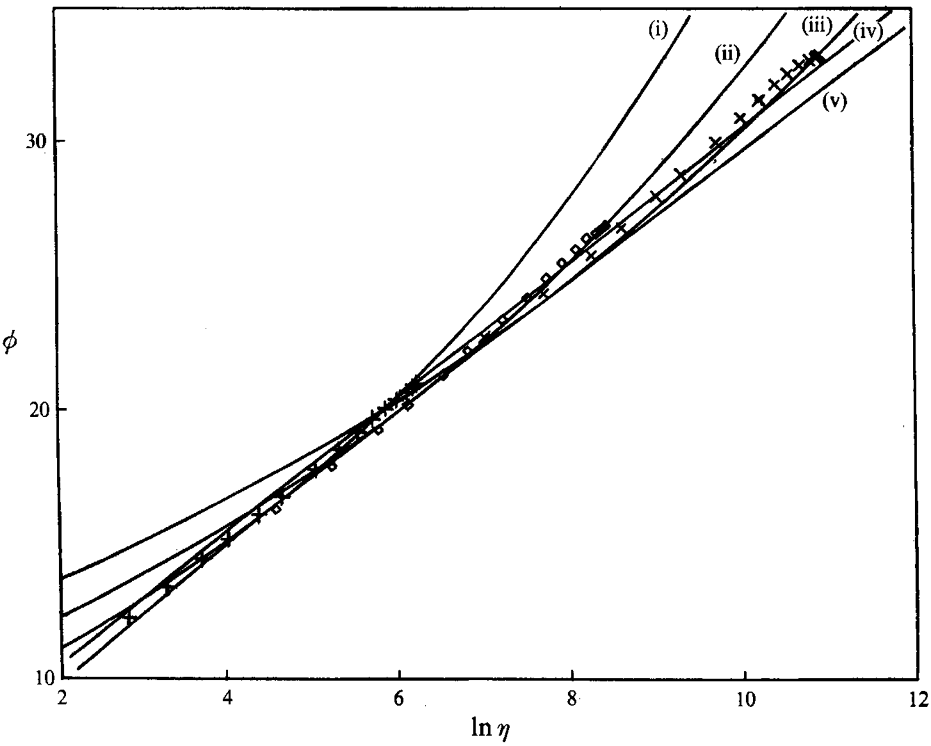

FIGURE 4. The experimental points, and the scaling law curves: curve (i): ($+$, $Re = 1.67 \times 10^4$); curve (ii): ($\diamond$, $Re = 2.05 \times 10^5$); curve (iii): ($\times$, $Re = 3.24 \times 10^6$); together with the curve of the universal logarithmic law (curve iv), and the envelope of scaling laws (curve v).

Number of experiment	σ_u	σ_s
1	0.042	0.019
2	0.030	0.011
3	0.021	0.010
4	0.015	0.007
5	0.016	0.012
6	0.014	0.007
7	0.023	0.010
8	0.019	0.009
9	0.017	0.004
10	0.012	0.007
11	0.031	0.020
12	0.015	0.005
13	0.013	0.003
14	0.012	0.007
15	0.013	0.014
16	0.014	0.015

TABLE 2. The relative mean square errors for the universal law (σ_u) and the proposed scaling law (σ_s).

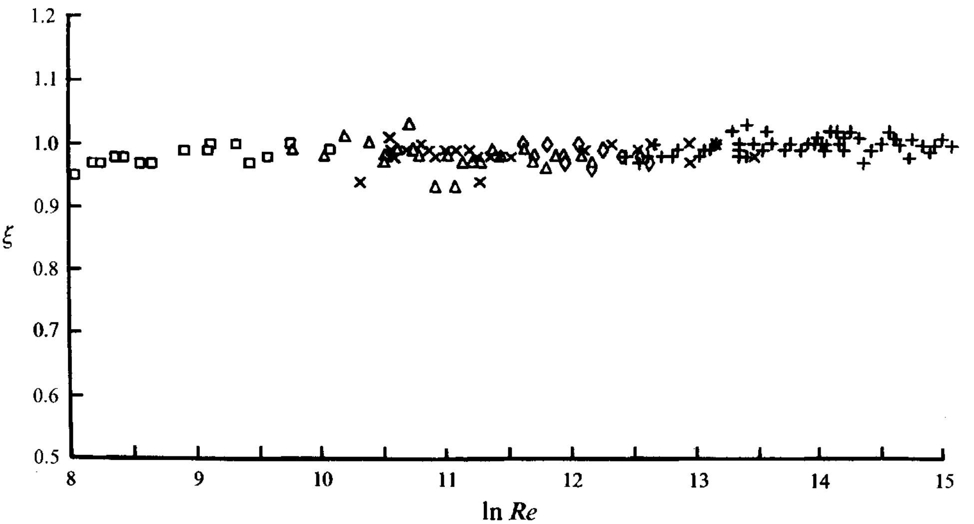

FIGURE 5. The experimental data for various tubes and various Reynolds number confirms the skin friction law (11)–(12) which follows from scaling law (9) with rather good accuracy: $\square$, $d = 1$ cm; $\triangle$, $d = 2$ cm; $\diamond$, $d = 3$ cm; $\times$, $d = 5$ cm; $+$, $d = 10$ cm.

By using this tube, he could apparently obtain results for Reynolds numbers which were smaller than those which he presented in his table 9 for the drag coefficients ($Re \geqslant 7.25 \times 10^5$). For these flows, the deviation from the scaling law on the one hand and the universal logarithmic law and envelope on the other should be larger, and better observable. Indeed, other data exist which demonstrate clearly a marked deviation of the experimental data upwards from the universal logarithmic law curve (see Monin & Yaglom 1971, p. 273).

2.5. *The skin-friction law*

As was shown in Part 1, the skin-friction law corresponding to the scaling law (9) for the velocity distribution assumes the form

$$\lambda = 8/\Psi^{2/(1+\alpha)}, \tag{11}$$

where

$$\Psi = \frac{e^{\frac{3}{2}}(\surd 3 + 5\alpha)}{2^{\alpha}\alpha(1+\alpha)(2+\alpha)}, \quad \alpha = \frac{3}{2\ln Re}. \tag{12}$$

It is convenient to compare the prediction (11)–(12) with experimental data (Nikuradze 1932, table 9) directly. The predictions and the data coincide to a very high accuracy. Therefore another form of comparison was chosen:

$$\xi = \lambda_e/\lambda = \tfrac{1}{8}\lambda_e \Psi^{2/(1+\alpha)} \tag{13}$$

was plotted as a function of $\ln Re$, denoting the experimentally determined value. In table 9 of Nikuradze (1932), a total of 125 points are available, corresponding to various Reynolds numbers ranging widely from slightly supercritical ($Re = 3.07 \times 10^3$) to very large ($Re = 3.23 \times 10^6$). All these results are presented in figure 5. Ideally, the quantity ξ should be equal to unity. We see in figure 5 that nearly all the deviations settle down to unity within the experimental scatter range.

3. Conclusions

The processing of Nikuradze's (1932) data as performed here allowed us to determine the necessary constants in the proposed scaling law, and confirmed it to high accuracy.

This verification by no means terminates the experimental checking of the scaling law (9). Now, with all the constants determined from Nikuradze's (1932) experiments, the scaling laws should be checked against other data, preferably further from the envelope of the scaling laws (developed turbulent flows in large tubes with smaller Reynolds number).

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